Relaxation process in a regime of quantum chaos

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We show that the quantum relaxation process in a classically chaotic open dynamical system is characterized by a quantum relaxation time scale t_q. This scale is much shorter than the Heisenberg time and much larger than the Ehrenfest time: $t_q \propto g^\alpha$ where g is the conductance of the system and the exponent α is close to $\frac{1}{2}$. As a result, quantum and classical decay probabilities remain close up to values $P \sim \exp(-\sqrt{g})$ similarly to the case of open disordered systems. [S1063-651X(97)51712-0]

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Recently there has been considerable interest in the statistical properties of the poles of the S matrix in mesoscopic quantum dots coupled to conducting metallic leads [1,2]. The statistical properties of these poles determine the effective lifetime of particles inside the dot and, therefore, are directly related to the conductance fluctuations and current relaxation inside the dot. In fact, the problem of current relaxation in diffusive mesoscopic samples connected to leads was addressed long ago [3]. Recently, interest in this problem was renewed and new effective methods based on the supersymmetry approach have been developed to study the problem in more detail [4]. For quasi-one-dimensional metallic samples the results of [3,4] predict that the current in the sample, being proportional to the probability $P(t)$ to stay inside the sample, will decay, up to a very long time, in an exponential way according to the classical solution of the diffusive equation, which describes the electron dynamics in disordered metallic samples: $P(t) \sim \exp(-t/t_c)$. Here $t_c \sim t_D = N^2/D$ is the diffusion time for a system of size N with diffusion coefficient D.

According to Refs. [3,4] the strong deviation of quantum probability P_q from its classical value P takes place only for $t > t_H$ where the quantum probability decays as $P_q(t) \sim \exp[-g \ln^2(t/t_H)]$. Here, $t_H = 1/\Delta (\hbar = 1)$ is the Heisenberg time, Δ is the level spacing inside the sample and $g = t_H/t_c = E_c/\Delta$ is the conductance of the sample with Thouless energy $E_c = 1/t_c$. At time t_H, $\ln P_q(t_H)/\ln P(t_H) \sim 2$. As was pointed out recently [5], less strong deviations $\{\ln[P_q(t_q)/P(t_q)] \sim 1\}$ should take place at a shorter time $t_q \sim t_c \sqrt{g}$ due to weak localization corrections according to equations obtained in Ref. [4]. Up to now these theoretical predictions for open systems have not been checked neither by numerical computations nor by laboratory experiments. Also the above results are based on an ensemble averaging over disorder and their validity for a quantum *dynamical* system that has *one fixed* classical limit is not evident. The investigation of this problem is also interesting from the viewpoint of semiclassical correspondence in a regime with exponentially fast spreading of narrow wave packets due to which the Ehrenfest time scale [6] is very short: $t_E \sim \ln N/\Lambda$, where Λ is the Liapunov exponent.

In this paper we study the quantum relaxation process in a dynamical model of quantum chaos where diffusion is caused by the underlying classical chaotic dynamics. This model, introduced in Ref. [7], describes a kicked rotator with absorbing boundary conditions (when the momentum is larger than some critical value). This open system can be considered as a model of light trapped in a small liquid droplet with a deformed boundary in which the rays, with orbital momentum less than some critical value, escape from the droplet because the refraction angle exceeds the critical value [8].

Contrary to the standard kicked rotator model [6] in which the matrix of the evolution operator is unitary, the absorption breaks the unitarity of the evolution matrix so that all eigenvalues move inside the unit circle. In other words, each eigenvalue can be written in the form $\lambda = e^{-i\epsilon} = \exp(-iE - \Gamma/2)$, where Γ characterizes the decay rate of the eigenstate. In this way absorption corresponds to ideal leads without reflections back to the sample. A similar approach, in which coupling to the continuum was studied on the basis of non-Hermitian Hamiltonians, has been developed and widely used by Weidenmüller *et al.* (see, for example, Ref. [9]).

In our model the quantum evolution of the wave function is described by the following quantum map:

$$\overline{\psi} = \hat{U}\psi = \hat{\mathcal{P}} e^{-iT\hat{n}^2/4} e^{-ik\cos\hat{\theta}} e^{-iT\hat{n}^2/4} \psi, \qquad (1)$$

where $\hat{\mathcal{P}}$ is a projection operator over quantum states n in the interval $(-N/2, N/2)$. Here, the commutator is $[\hat{n}, \hat{\theta}] = -i$ and the classical limit corresponds to $k \to \infty$, $T \to 0$ while the classical chaos parameter $K = kT$ remains constant. In the classical limit the dynamics is described by the Chirikov standard map [6]:

$$\overline{n} = n + k\sin\left[\theta + \frac{Tn}{2}\right], \quad \overline{\theta} = \theta + \frac{T}{2}(n + \overline{n}). \qquad (2)$$

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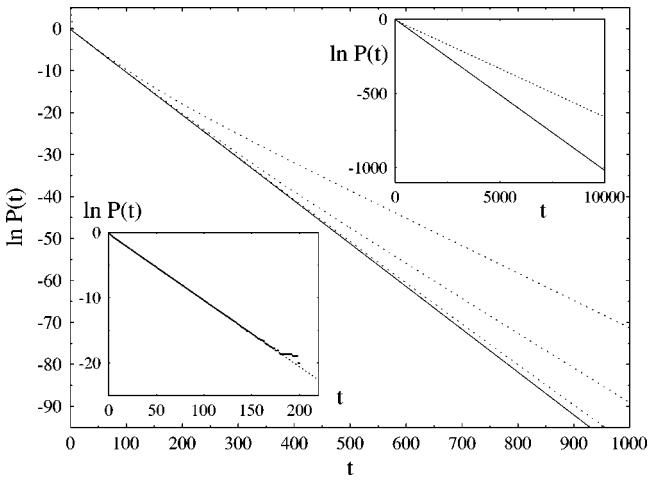

FIG. 1. Classical and quantum probability decay for $K=7$ and $N/k=4$. The full line shows the fit to the classical decay. Dotted lines give the quantum probability $P_q(t)$ for $N=5001,\ 20\,001,\ 130\,001$ (upper, middle, and lower curves, respectively). The lower insert shows the classical probability actually computed from $M=9\times10^9$ orbits and the fit is shown by the dotted line. The upper insert shows the classical (full line) and the quantum (dotted line) asymptotic decay for $N=5001$. Time t is given in number of kicks.

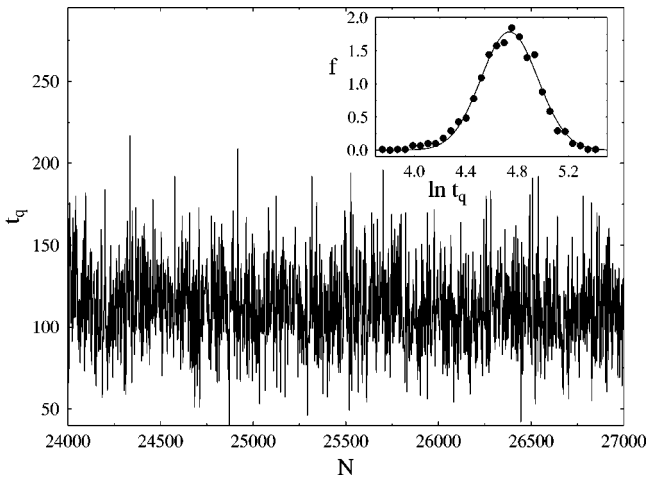

FIG. 2. Mesoscopic fluctuations of the quantum relaxation time t_q for different system sizes N. The insert shows the statistical distribution of fluctuations f, which is close to a log normal distribution of width 0.22. Time t is given in number of kicks.

For the classical computations, in analogy with the quantum model, all classical trajectories escaped from the interval $(-N/2,N/2)$ are absorbed and never return back. Due to this absorption, in the regime of strong chaos ($K\gg1$) with one chaotic component (no islands of stability), the classical probability to stay inside the interval $(-N/2,N/2)$ decays exponentially with time: $P(t)\sim\exp(-\gamma_c t)$. The time scale $t_c=1/\gamma_c\sim t_D$ is determined by the diffusion time t_D required to reach the absorbing boundary from the center. Since the diffusion rate is $D=\langle\Delta n^2\rangle/\Delta t\sim k^2/2$ then $\gamma_c=E_c\sim1/t_D=k^2/N^2$. In order to study the quantum relaxation we fixed the classical chaos parameter $K=7$ and the ratio $N/k=4$. In this way the diffusion time t_D is constant when $N\to\infty$ and this allows us to investigate the semiclassical behavior. Moreover, $t_D\gg1$, which justifies the diffusive approximation. We note also that the system (1) with $-N/2<n<N/2$ and $\hat{\mathcal{P}}=\hat{\mathbb{1}}$ coupled to open leads ($T=0$ for $|n|>N/2$) had been studied in Ref. [10]. The results obtained there showed that this model has universal conductance fluctuations [11] and other properties very similar to mesoscopic metallic samples.

The numerical solution of the classical problem was obtained by the iterating map (2) for $M=9\times10^9$ different initial conditions homogeneously distributed on the line $n=0$. The results demonstrate a clear exponential decay $P(t)=\exp(-\gamma_c t-b)$ with $\gamma_c=0.101882(1), b=0.17774(5)$ (see Fig. 1). This exponential decay shows that for $K=7$ the phase space is completely chaotic without any island of stability. Even with such a high number of orbits, the classical computations allow us to obtain *directly* the probability $P(t)$ with 10% accuracy only up to the level $\overline{P}\approx5\times10^{-8}$ ($t\approx165$). This limitation is due to statistical errors appearing for a finite number of trajectories. In spite of this, the decay rate γ_c can be found with very high precision, which allows us to extrapolate the probability behavior to larger times.

For the quantum evolution we choose the corresponding initial condition in which only the level $n=0$ is populated and we studied numerically the quantum dynamics (1) for different N. We have found that the quantum probability $P_q(t)$ follows the classical one up to a time t_q, after which it starts to decay at a slower rate (Fig. 1). We determined the quantum relaxation time t_q by the condition $\ln[P_q(t_q)/P(t_q)]=0.1$, which corresponds to 10% deviation. The comparison of quantum and classical probabilities is shown in Fig. 1. The values of t_q, obtained in this way, strongly fluctuate with changing the system size N as is typical for mesoscopic systems (Fig. 2). These fluctuations are satisfactory described by a log normal distribution (insert in Fig. 2), but a more detailed analysis is required to determine precisely their statistical properties.

To suppress the fluctuations, we average $\ln P_q(t)$ over different system sizes by changing N in a small interval δN around a given N. Typically we averaged up to 500 different N values. This allows us to determine the averaged ratio $R(t)=\langle\ln[P_q(t)/P(t)]\rangle$ of quantum to classical probability. For all these N values the classical dynamics is *exactly the same* since we kept $K=7$ and $N/k=4$. Then the quantum relaxation time t_q (averaged) at the 10% level is determined by the condition $R(t_q)=0.1$. The dependence of $R(t)$ on time, for different N, is shown in Fig. 3. It is clearly seen that t_q grows as we approach the semiclassical limit $N\to\infty$. The rescaling of data for $R(t)$ as a function of t/t_q shows a satisfactory global scaling behavior of quantum probability (see insert in Fig. 3). The dependence of t_q on N is shown in Fig. 4 for the semiclassical regime $500<N\leqslant130\,001$. This regime corresponds to a variation of conductance $g=N/t_c$ in the interval $50<g\leqslant13\,000$. The fit of numerical data gives a power-law dependence $t_q\approx0.19t_cN^{\alpha}$ with $\alpha\approx0.41$. This power remains the same for the 5% deviation level (Fig. 4).

Here we propose a qualitative explanation of α value based on the fact that in the open system the physics is affected not by the level spacing Δ but by the distribution of poles of the scattering matrix S that describes the coupling to the leads. These poles are located in the complex energy plane and their imaginary parts determine the decay probability of eigenmodes inside the sample. For our model, the

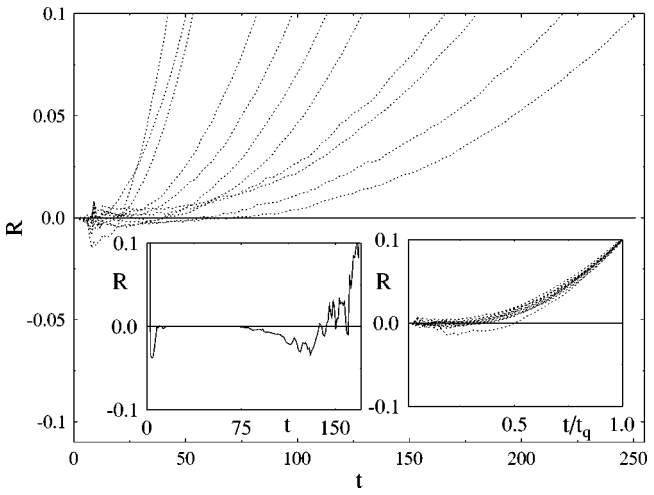

FIG. 3. The average ratio $R(t)=\langle \ln[P_q(t)/P(t)]\rangle$ as a function of time for different N from $N=2500$ (left curve) to $N=129\,500$ (right curve). The horizontal full line corresponds to $P_q(t)=P(t)$. The probability $P(t)$ is given by numerical data obtained with $M=9\times10^9$ orbits for $t\leqslant 70$ and by the fit from Fig. 1 (see text) for $t>70$. The left insert shows the ratio of the numerically computed classical probability $P(t)$ to the fit function. The deviations from the fit, for $t>70$, are due to statistical errors related to finite M. The right insert demonstrates the scaling behavior of $P_q(t)$ on the variable t/t_q, where the t_q values are determined by condition $R(t_q)=0.1$. Time t is given in number of kicks.

poles are simply given by the evolution operator (1). The eigenvalues of $\hat{U}$ are distributed in a narrow ring of width E_c inside the unitary circle [7]. This is typical for diffusive samples coupled to strongly absorbing leads. As a result, N complex eigenvalues are homogeneously distributed in a ring of total area $A\approx E_c$ and the distance between them, in the complex plane, is $\delta\approx\sqrt{E_c/N}$. In the classical limit this spacing goes to zero and one obtains a continuous density of poles. However, for finite N, the separation of poles is finite and can be resolved after a time $t_q\sim 1/\delta$. According to this argument, which is independent of the symmetry and dimensionality of the problem, the deviation between quantum and classical probabilities will take place at

$$t_q\approx 0.38\sqrt{t_c N}=0.38 t_c\sqrt{g}, \tag{3}$$

where the numerical coefficient has been extracted from Fig. 4. The theoretical dependence (3), which corresponds to $\alpha=1/2$, is different but close to the numerical value $\alpha=0.41$. We attribute this difference to a not sufficiently large value of $\sqrt{g}$. Indeed, neglecting the values with $\sqrt{g}<15$, we obtain $\alpha=0.44$ for 10% deviation and $\alpha=0.45$ for 5% deviation; these values are closer to the theoretical prediction $\alpha=0.5$.

The scale t_q can be also explained in a more standard way based on weak-localization corrections [5,4]. Indeed, the quantum interference gives a decrease of the diffusion rate $1/t_c\propto D\to D(1-at/t_H)$, where a is some constant (diffusion stops at time t_H). As a result $\ln(P_q/P)\approx at_q^2/(t_c t_H)\sim 1$ and one gets Eq. (3).

For very large times, the decay of $P_q(t)$ is determined by the eigenvalue $\epsilon=E-i\Gamma/2$ with minimal $\Gamma=\Gamma_{\min}$. This asymptotic behavior should start from some time scale t_f, which can be estimated in a following way. Similarly to the results obtained for complex matrices [12], the eigenvalues ϵ should be distributed in a region with a smooth boundary in the complex plane with approximately constant density of points for $\Gamma\sim\gamma_c$. Typically this boundary is parabolic near the extremal Γ_0 close to $\Gamma_{\min}$. Due to this, the relative number of eigenvalues dW in the interval $d\Gamma$ is given by $dW/d\Gamma\sim\sqrt{(\Gamma-\Gamma_0)}/\gamma_c^{3/2}$. The total probability in the interval $\delta\Gamma=\Gamma-\Gamma_0$ is $W\sim(\delta\Gamma/\gamma_c)^{3/2}$. The distance between the two lowest values of Γ can be estimated from the condition $W\sim 1/N$, which gives $\delta\Gamma\sim\gamma_c N^{-2/3}$. Then $t_f\approx 0.4 t_c N^{2/3}$, which is much larger than t_q. The numerical factor here was determined from the two lowest values of Γ for $N=5001$: $\Gamma_{\min}=\Gamma_1=0.065\,309$ and nearest $\Gamma_2=0.066\,203$. These values were obtained by direct diagonalization of the matrix U. The rate of the asymptotic decay of $P_q(t)$ for $N=5001$ (Fig. 1) agrees, up to six digits accuracy, with the above value of $\Gamma_{\min}$. We note that a typical size of fluctuations for poles is δ and so we expect that $\gamma_c-\Gamma_{\min}\sim\delta\sim 1/t_q$. Since in our model $t_f\ll t_H$ and the classical limit is fixed we do not see the quantum deviations discussed in Refs. [3,4] for $t>t_H$. At large g one should average over an exponentially large number of realizations to observe them.

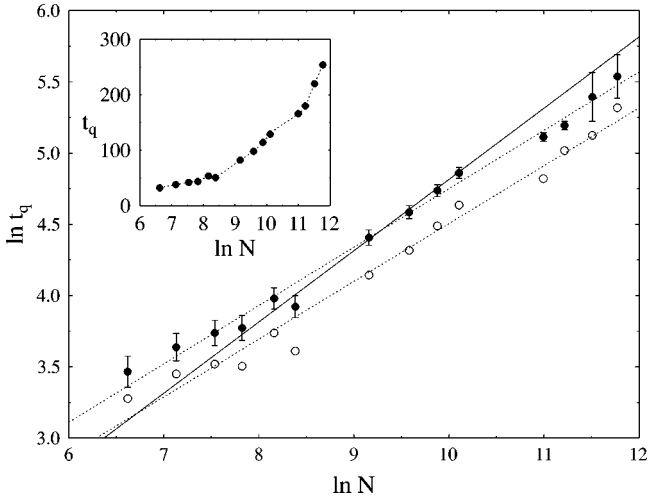

FIG. 4. Dependence of the quantum relaxation time t_q on the system size $N=gt_c$ in logarithmic scale. Points refer to 10% deviation level ($R=0.1$) while circles refer to 5% deviation ($R=0.05$). The two dotted lines give the fit $t_q=1.9N^{0.41}$ and $t_q=1.5N^{0.41}$, respectively. The full line gives the theoretical prediction with $\alpha=1/2$ (3). The insert shows the 10% data in a semilog scale. Time t is given in number of kicks.

The largest value of t_q we have numerically obtained (at $N=1.3\times10^5$) is $t_q=254$, which corresponds to a probability $P_q(t_q)\approx 5\times10^{-12}$. On the other hand, the classical simulation with $M=9\times10^9$ orbits allows us to directly compute the classical $P(t)$ with 10% accuracy only up to $t\approx 165$, which corresponds to a probability $\overline{P}\approx 5\times10^{-8}$. In order to reach the above level of quantum accuracy one should iterate $M\approx10^{14}$ orbits, which is already beyond the capability of present day computers. Moreover, the value of P_q at 10% accuracy level can be easily increased by orders of magnitude since, according to Eq. (3) t_q grows proportionally to $\sqrt{N}$. Instead, for classical computations, the number of required orbits M will increase exponentially ($\overline{P}\sim 1/M$). This

demonstrates that quantum computations of exponential relaxation processes are much more efficient than classical simulations with a large number of orbits. The possibility to efficiently compute the quantum probability $P_q(t)$ up to very long times $t_q \gg 1/\Lambda$ allows us to numerically estimate the measure of the integrable component μ in the phase space of the classical system. Indeed, for $\mu > 0$, the existence of integrable islands leads to an asymptotic power-law decay of correlations $P(t) \propto t^{-0.5}$ [13]. Since in our numerical data the quantum probability decays exponentially up to $P_q(t_q) \approx 5 \times 10^{-12}$ we assume that the measure of the integrable component is $\mu < P_q(t_q) \approx 5 \times 10^{-12} \times (1 \pm 0.1)$, being much smaller than the relative size of quantum cell $1/N$. Here, the error bar gives the average fluctuation of $P_q(t_q = 254)$ obtained for 77 values of N.

Also, it is interesting to note that the Ehrenfest time scale t_E is much smaller than the quantum relaxation time t_q: $t_E/t_q \sim \ln N/\sqrt{N} \ll 1$. For example, for $N = 1.3 \times 10^5$ we have $t_q = 254$ while $t_E = \ln N/\Lambda \approx 9.4$ ($\Lambda \approx \ln K/2 \approx 1.25$). This shows that the agreement between quantum and classical relaxation continues for a time scale that is much larger than the time of wave packet spreading. However, for $t > t_E$ there is no exponential instability in the quantum motion [6,14]. As a result, correlation functions of the type $C(\tau) = \langle \sin\theta(t) \sin\theta(t+\tau) \rangle$, which, in the regime of strong chaos, decay exponentially in the classical case ($\ln|C| \sim -\Lambda\tau$), in the quantum case decay only during the Ehrenfest time scale up to $\ln|C| \sim -\ln N$ ($t_E \ll \tau \ll t_c$). This is similar to what happens in closed (unitary) systems such as the kicked rotator [14]. This example shows that exponential relaxation is not necessarily related to exponential local instability and positive Kolmogorov-Sinai entropy.

Recently the scale t_q has been obtained on the basis of random matrix theory and supersymmetry for kicked rotator with random phases [15]. The related results for prelocalized states in closed systems were discussed in Ref. [16].

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